Warsaw University of Technology

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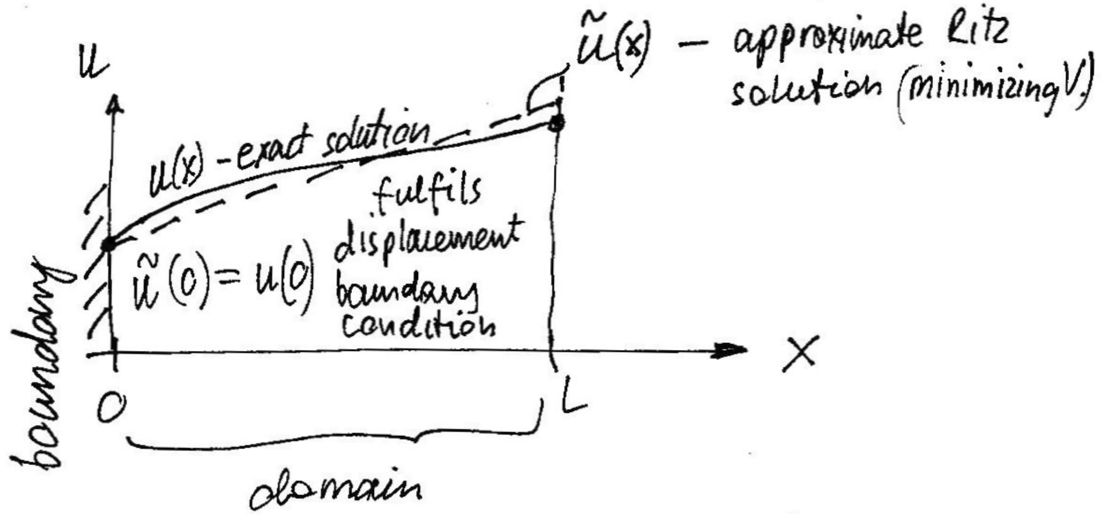
Finite element method (FEM)

Ritz method

04.2021

RITZ METHOD

Ritz method is one of the approximate methods based on the principle of minimum total potential energy and a global approximation.



In this method the approximate function $\tilde{u}(x)$ is assumed for the entire domain as:

$$\tilde{u}(x) = \sum_{i=1}^{n} c_i \cdot f_i(x) \quad \leftarrow \text{global approximation}$$

c_i – unknown constants; $f_i(x)$ – functions for example:

$$f_i = x^{i-1}\,;\; f_1 = 1\,;\; f_2 = x\,;\; f_3 = x^2\,;\; f_4 = x^3\,;\; f_5 = x^4\,;\ldots$$

We use a global approximation when the approximate function $\tilde{u}(x)$ is used for the entire domain.

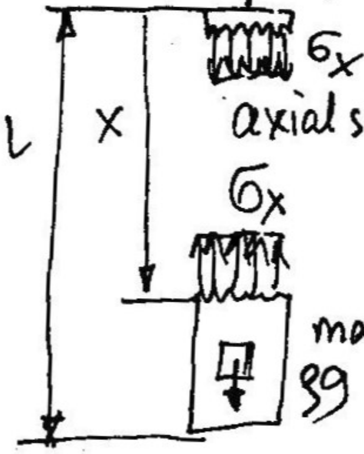
Example. Find displacement, strain and stress in a bar loaded by gravity g. Use the Ritz method $(i=2,3,4)$.
Compare the approximate solution with an exact solution.

E - Young's modulus
A - crossection
ρ - density
l - lenght

(linear and isotropic material)

Exact solution

$N(0) = \sigma_x(0) \cdot A = m \cdot g$ (reaction)



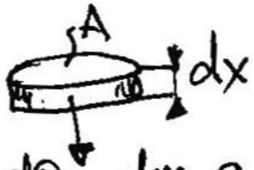
equilibrium equation:

$$-\sigma_x \cdot A + \int_x^l \rho g A dx = 0$$

$$\sigma_x = \rho g (l - x)$$

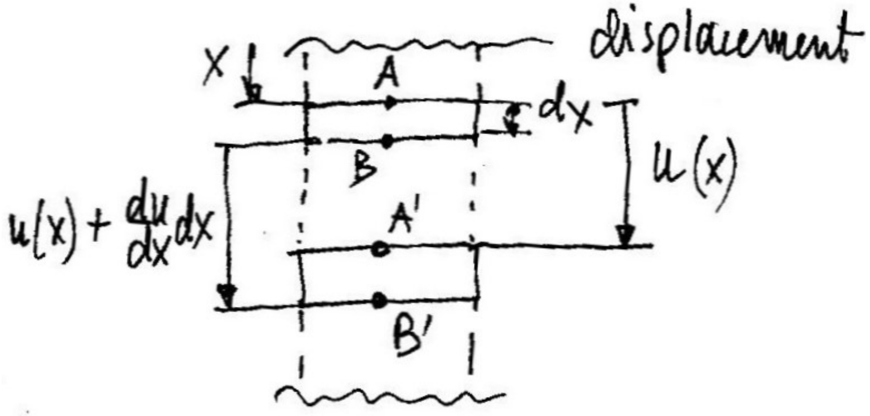
$$\sigma_x(0) = \rho g l$$

$$\sigma_x(l) = 0$$



$$dQ = dm \cdot g = dV \cdot \rho g = \rho g A dx$$

traction: $\frac{dQ}{dx} = \rho g A$



$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{\rho g (l - x)}{E}$$

$$\varepsilon_x(0) = \frac{\rho g l}{E}, \quad \varepsilon_x(l) = 0$$

$$\varepsilon_x = \frac{(A'B') - AB}{AB} = \frac{dx + u(x) + \frac{du}{dx}dx - u(x) - dx}{dx} =$$

$$= \frac{du}{dx} \Rightarrow$$

$$u(x) = \int \varepsilon_x dx + \underset{\substack{\parallel \\ 0}}{u(0)} = \int \frac{\rho g (l-x)}{E} dx = \frac{\rho g}{E}\left(lx - \frac{x^2}{2}\right)$$

$$u(0) = 0, \quad u\left(\frac{l}{2}\right) = \frac{\rho g}{E}\left(\frac{l^2}{2} - \frac{l^2}{4 \cdot 2}\right) = \frac{3 \rho g l^2}{8E}, \quad u(l) = \frac{\rho g l^2}{2E}$$

1°) $n = 2$, $\tilde{u}(x) = c_1 \cdot f_1 + c_2 f_2 = c_1 \cdot 1 + c_2 \cdot x$

boundary condition $\tilde{u}(0) = 0 \Rightarrow c_1 + c_2 \cdot 0 = 0 \Rightarrow c_1 = 0$

$\tilde{u}(x) = c_2 \cdot x$, $\tilde{\varepsilon}_x = \frac{d\tilde{u}}{dx} = c_2$, $\tilde{\sigma}_x = E\tilde{\varepsilon}_x = E \cdot c_2$

$$U = \frac{1}{2} \int_{\Omega} \left(\tilde{\sigma}_x \tilde{\varepsilon}_x + 0 \cdot \tilde{\varepsilon}_y + 0 \cdot \tilde{\varepsilon}_z + 0 \cdot 0 + 0 \cdot 0 + 0 \cdot 0 \right) d\Omega =$$

$$= \frac{1}{2} \int_0^L \tilde{\sigma}_x \tilde{\varepsilon}_x \int_A dA \, dx = \frac{1}{2} \int_0^L E c_2 \cdot c_2 \cdot A \, dx = \frac{EA}{2} c_2^2 L$$

$$W = \int_{\Omega} \underset{1\times 3}{\lfloor X \rfloor} \cdot \underset{3\times 1}{\{u\}} d\Omega + \underbrace{\int_{\Gamma} \underset{1\times 3}{\lfloor p \rfloor} \cdot \underset{3\times 1}{\{u\}} d\Gamma}_{0 \text{ (no surface load)}} = \int_{\Omega} (X\tilde{u} + 0\tilde{v} + 0\tilde{w}) d\Omega =$$

$$= \int_0^L \rho g \tilde{u} \int_A dA \, dx = \int_0^L \rho g A \tilde{u} \, dx = \int_0^L \rho g A c_2 x \, dx = \frac{\rho g A L^2}{2} c_2$$

$$V = \frac{EA}{2} c_2^2 l - \frac{\rho g A l^2}{2} c_2 \quad ; \quad V \to \min$$

$$\frac{\partial V}{\partial c_2} = 0 \quad \Rightarrow \quad 2\frac{EA}{2} c_2 \cdot l - \rho g \frac{A l^2}{2} = 0$$

$$c_2 = \frac{\rho g A l^2}{2EAl} = \frac{\rho g l}{2E}$$

$$\tilde{u}(x) = \frac{\rho g l}{2E} x \quad ; \quad \tilde{u}(0) = 0 \quad ; \quad \tilde{u}(l) = \frac{\rho g l^2}{2E}$$

$$\tilde{u}(x) \neq u(x)$$

$$\tilde{u}(l) = u(l)$$

$$\tilde{\varepsilon}_x = \frac{\rho g l}{2E} = \text{const} = \frac{\varepsilon_x(0) + \varepsilon_x(l)}{2}$$

$$\tilde{\sigma}_x = \frac{\rho g l}{2} = \text{const} = \frac{\sigma_x(0) + \sigma_x(l)}{2}$$

$2°)\ n=3;\quad \tilde{u}(x)=c_1f_1+c_2f_2+c_3f_3=c_1 1+c_2\cdot x+c_3\cdot x^2$

boundary condition: $\tilde{u}(0)=0 \Rightarrow c_1+c_2\cdot 0+c_3\cdot 0^2=0 \Rightarrow c_1=0$

$\tilde{u}(x)=c_2x+c_3x^2,\quad \tilde{\varepsilon}_x=\frac{d\tilde{u}}{dx}=c_2+2c_3x,\quad \tilde{\sigma}_x=E\cdot\tilde{\varepsilon}_x$

$$U=\frac{1}{2}\int_0^l \tilde{\sigma}_x\tilde{\varepsilon}_x A\,dx=\frac{EA}{2}\int_0^l (c_2+2c_3x)^2dx=$$

$$=\frac{EA}{2}\int_0^l \left(c_2^2+4c_2c_3x+4c_3^2x^2\right)dx=$$

$$=\frac{EA}{2}\left(c_2^2x+2c_2c_3x^2+\frac{4}{3}c_3^2x^3\right)\Big|_0^l=\frac{EA}{2}\left(l\cdot c_2^2+2l^2c_2c_3+\frac{4}{3}l^3c_3^2\right)$$

$$W=\int_0^l \rho gA\tilde{u}\,dx=\rho gA\int_0^l (c_2x+c_3x^2)dx=\rho gA\left(\frac{c_2x^2}{2}-\frac{c_3}{3}x^3\right)\Big|_0^l=$$

$$=\rho gA\left(\frac{l^2}{2}c_2-\frac{l^3}{3}c_3\right),\quad V\to min$$

$$\begin{cases} \dfrac{\partial V}{\partial c_2} = 0 \\ \dfrac{\partial V}{\partial c_3} = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial U}{\partial c_2} - \dfrac{\partial W}{\partial c_2} = 0 \\ \dfrac{\partial U}{\partial c_3} - \dfrac{\partial W}{\partial c_3} = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{EA}{2}\left(2lc_2 + 2l^2 c_3\right) - \dfrac{\rho g A l^2}{2} = 0 \\ \dfrac{EA}{2}\left(2l^2 c_2 + \dfrac{8l^3}{3} c_3\right) - \dfrac{\rho g A l^3}{3} = 0 \end{cases}$$

$$\begin{matrix} \left(\cdot\frac{1}{l}\right) \\ \left(\cdot\frac{1}{l^2}\right) \end{matrix} \begin{cases} 2lc_2 + 2l^2 c_3 = \dfrac{\rho g l^2}{E} \\ 2l^2 c_2 + \dfrac{8l^3}{3} c_3 = \dfrac{2\rho g l^2}{3E} \end{cases} \Rightarrow \begin{cases} 2c_2 + 2lc_3 = \dfrac{\rho g l}{E} \\ 2c_2 + \dfrac{8}{3} l c_3 = \dfrac{2\rho g l}{3E} \end{cases}$$

$$① - ② : \; -\frac{2}{3} l c_3 = \frac{1}{3}\frac{\rho g l}{E} \Rightarrow c_3 = -\frac{\rho g}{2E}, \; c_2 = \frac{\rho g l}{2E} - l c_3 = \frac{\rho g l}{E}$$

$$\tilde{u}(x) = \frac{\rho g l}{E} x - \frac{\rho g}{2E} x^2 = \frac{\rho g}{E}\left(lx - \frac{x^2}{2}\right) = u(x)$$

$$\tilde{\varepsilon}_x(x) = \varepsilon_x, \quad \tilde{\sigma}_x = \sigma_x$$

3°) $n=4$, $\tilde{u}(x)=c_1 f_1+c_2 f_2+c_3 f_3+c_4 f_4=c_1\cdot 1+c_2 x+c_3 x^2+c_4 x^3$

boundary condition $\tilde{u}(0)=0 \Rightarrow c_1+0+0+0=0 \Rightarrow c_1=0$

$\tilde{u}(x)=c_2 x+c_3 x^2+c_4 x^3$, $\tilde{\varepsilon}_x=\frac{d\tilde{u}}{dx}=c_2+2c_3 x+3c_4 x^2$, $\tilde{\sigma}_x=E\tilde{\varepsilon}_x$

$$U=\frac{EA}{2}\int_0^l (c_2+2c_3 x+3c_4 x^2)^2\,dx$$

$$V=U-W\rightarrow \min$$

$$W=\rho g A\int_0^l (c_2 x+c_3 x^2+c_4 x^3)\,dx$$

$$\frac{\partial V}{\partial c_2}=0,\ \frac{\partial V}{\partial c_3}=0,\ \frac{\partial V}{\partial c_4}=0$$

⋮

$$c_2=\frac{\rho g l}{E},\quad c_3=-\frac{\rho g}{2E},\quad c_4=0$$

$$\tilde{u}(x)=\frac{\rho g}{E}\left(lx-\frac{x^2}{2}\right)=u(x)$$

$$\tilde{\varepsilon}_x(x)=\varepsilon_x(x),\quad \tilde{\sigma}_x(x)=\sigma_x(x)$$

4°) $f_i = \sin\frac{i\pi x}{2l}$ $\quad i = 1, 2, \dots n$

$n = 2 \qquad \tilde{u}(x) = c_1 \cdot \sin\frac{\pi x}{2l} + c_2 \cdot \sin\frac{\pi x}{l}$

$$\tilde{\varepsilon}_x = \frac{d\tilde{u}}{dx} = \frac{c_1\pi}{2l} \cdot \cos\frac{\pi x}{2l} + \frac{c_2\pi}{l} \cdot \cos\frac{\pi x}{l}, \qquad \tilde{\sigma}_x = E \cdot \tilde{\varepsilon}_x$$

$$U = \frac{1}{2}\int_{\Omega} \tilde{\sigma}_x \tilde{\varepsilon}_x \, d\Omega = \frac{EA}{2}\int_0^l \tilde{\varepsilon}_x^2 \, dx = \frac{EA}{2}\int_0^l \left(\frac{c_1\pi}{2l}\cos\frac{\pi x}{2l} + \frac{c_2\pi}{l}\cos\frac{\pi x}{l}\right)^2 dx =$$

$$= \frac{EA}{2}\int_0^l \left(\frac{c_1^2\pi^2}{4l^2} \cdot \cos^2\frac{\pi x}{2l} + \frac{2c_1c_2\pi^2}{2l^2}\cos\frac{\pi x}{2l} \cdot \cos\frac{\pi x}{l} + \frac{c_2^2\pi^2}{l^2} \cdot \cos^2\frac{\pi x}{l}\right) dx \Rightarrow$$

$$\frac{EA}{2}\frac{c_1^2\pi^2}{4l^2}\int_0^l \cos^2\frac{\pi x}{2l}\,dx = \left|\begin{array}{l} y=\frac{\pi x}{2l} \\ dy=\frac{\pi}{2l}dx \\ x=0 \Rightarrow y=0 \\ x=l \Rightarrow y=\frac{\pi}{2} \end{array}\right| = \frac{EAc_1^2\pi}{4l}\int_0^{\frac{\pi}{2}} \cos^2 y\,dy =$$

$$= \left|\begin{array}{l} \text{by parts} \\ \int f\,dg = fg - \int g\,df \\ f=\cos y\ ,\ dg=\cos y\,dy \\ g=\int dg = \sin y \\ df = -\sin y\cdot dy \end{array}\right| = \frac{EAc_1^2\pi}{4l}\left(\underbrace{\cos y\sin y\Big|_0^{\frac{\pi}{2}}}_{=0} + \int_0^{\frac{\pi}{2}}\sin^2 y\,dy\right) =$$

$$= \frac{EAc_1^2\pi}{4l}\int_0^{\frac{\pi}{2}}\left(1-\cos^2 y\right)dy = \frac{EAc_1^2\pi^2}{8l} - \frac{EAc_1^2\pi}{4l}\int_0^{\frac{\pi}{2}}\cos^2 y\,dy$$

$$\Rightarrow \frac{EA}{2}\frac{c_1^2\pi^2}{4l^2}\int_0^l \cos^2\frac{\pi x}{2l}\,dx = \frac{1}{2}\frac{EAc_1^2\pi^2}{8l} = \frac{\pi^2 EA}{16l}c_1^2$$

$$\frac{EA}{2}\cdot\frac{2c_1c_2\pi^2}{2l^2}\int_0^l \cos\frac{\pi x}{2l}\cdot\cos\frac{\pi x}{l}\,dx = \left|\begin{array}{l}\cos\frac{\pi x}{2l}\cdot\cos\frac{\pi x}{l} = \\ =\frac{1}{2}\left(\cos\left(\frac{\pi x}{2l}-\frac{\pi x}{l}\right)+\cos\left(\frac{\pi x}{2l}+\frac{\pi x}{l}\right)\right)= \\ =\frac{1}{2}\left(\cos\frac{-\pi x}{2l}+\cos\frac{3\pi x}{2l}\right)\end{array}\right| =$$

$$= \frac{EAc_1c_2\pi^2}{2l^2}\cdot\left(\frac{1}{2}\int_0^l \cos\left(\frac{-\pi x}{2l}\right)dx + \frac{1}{2}\int_0^l \cos\frac{3\pi x}{2l}\,dx\right) =$$

$$= \frac{EAc_1c_2\pi^2}{4l^2}\cdot\left(-\frac{2l}{\pi}\cdot\sin\left(\frac{-\pi x}{2l}\right)\Big|_0^l + \frac{2l}{3\pi}\cdot\sin\frac{3\pi x}{2l}\Big|_0^l\right) =$$

$$= \frac{EAc_1c_2\pi^2}{4l^2}\left(-\frac{2l}{\pi}\cdot(-1)+\frac{2l}{3\pi}(-1)\right) = \frac{EAc_1c_2\pi^2}{4l}\cdot\frac{2}{3}\cdot\frac{2l}{\pi} = \frac{\pi EA}{3l}c_1\cdot c_2$$

$$\frac{EA}{2}\frac{c_2^{\,2}\pi^2}{l^2}\int_0^l \cos^2\frac{\pi x}{l}\,dx = \frac{\pi^2 EA}{4l}c_2^{\,2}$$

$$U = \frac{\pi^2 EA}{16l}c_1^{\,2} + \frac{\pi EA}{3l}c_1\cdot c_2 + \frac{\pi^2 EA}{4l}c_2^{\,2}$$

$$W = \int_{\Omega} \rho g \tilde{u}(x)\, d\Omega = \rho g A \int_0^L \left(c_1 \cdot \sin\frac{\pi x}{2L} + c_2 \cdot \sin\frac{\pi x}{L} \right) dx =$$

$$= \rho g A c_1 \int_0^L \sin\frac{\pi x}{2L}\, dx + \rho g A c_2 \int_0^L \sin\frac{\pi x}{L}\, dx =$$

$$= 2\rho g A c_1 L \left(-\cos\frac{\pi x}{2L} \right) \Big|_0^L + \rho g A c_2 \frac{L}{\pi} \left(-\cos\frac{\pi x}{L} \right) \Big|_0^L =$$

$$= \frac{2\rho g A L}{\pi} \cdot c_1 + \frac{2\rho g A L}{\pi} \cdot c_2$$

$$V = \frac{\pi^2 EA}{16l} c_1^2 + \frac{EA\pi}{3l} c_1 c_2 + \frac{\pi^2 EA}{4l} c_2^2 - \frac{2\rho g A l}{\pi}(c_1 + c_2)$$

$$\frac{\partial V}{\partial c_1} = 0 \quad \Rightarrow \begin{cases} \dfrac{\pi^2 EA}{8l} c_1 + \dfrac{EA\pi}{3l} c_2 - \dfrac{2\rho g A l}{\pi} = 0 \\ \dfrac{EA\pi}{3l} \cdot c_1 + \dfrac{\pi^2 EA}{2l} c_2 - \dfrac{2\rho g A l}{\pi} = 0 \end{cases}$$

$$\frac{\partial V}{\partial c_2} = 0 \quad \Rightarrow$$

$$\cdot \frac{l}{EA\pi}$$

$$\begin{bmatrix} \frac{\pi}{8} & \frac{1}{3} \\ \frac{1}{3} & \frac{\pi}{2} \end{bmatrix} \cdot \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{2\rho g l^2}{E\pi^2} \\ \dfrac{2\rho g l^2}{E\pi^2} \end{Bmatrix}$$

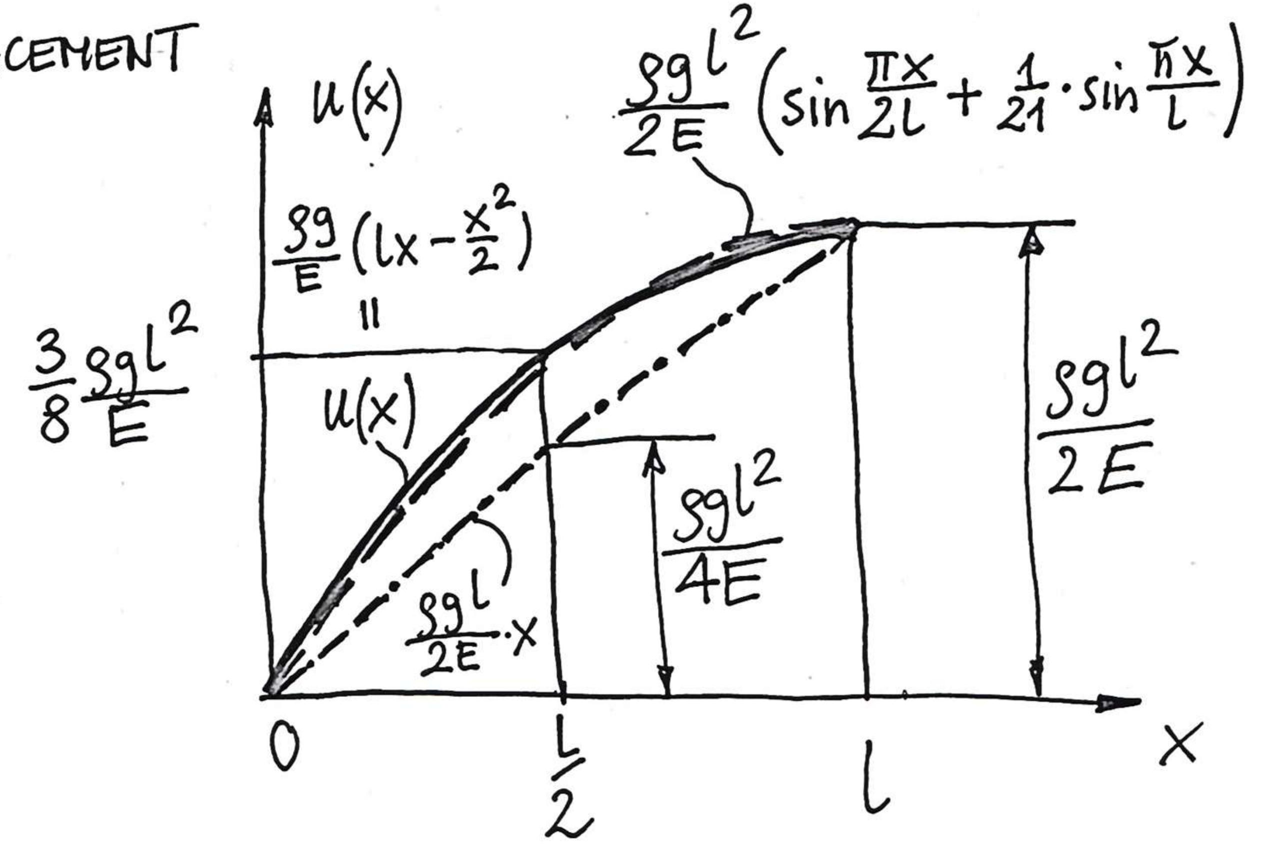
$$\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{bmatrix} \frac{\pi}{8} & \frac{1}{3} \\ \frac{1}{3} & \frac{\pi}{2} \end{bmatrix}^{-1} \cdot \frac{2\rho g l^2}{E\pi^2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$c_1 \cong \frac{\rho g l^2}{2E} \quad , \quad c_2 \cong \frac{\rho g l^2}{42E}$$

$$\tilde{u}(x) = \frac{\rho g l^2}{2E}\left(\sin\frac{\pi x}{2l} + \frac{1}{21}\sin\frac{\pi x}{l}\right)$$

$$\tilde{\varepsilon}_x(x) = \frac{d\tilde{u}}{dx} = \frac{\pi \rho g l}{4E}\left(\cos\frac{\pi x}{2l} + \frac{2}{21}\cos\frac{\pi x}{l}\right)$$

DISPLACEMENT

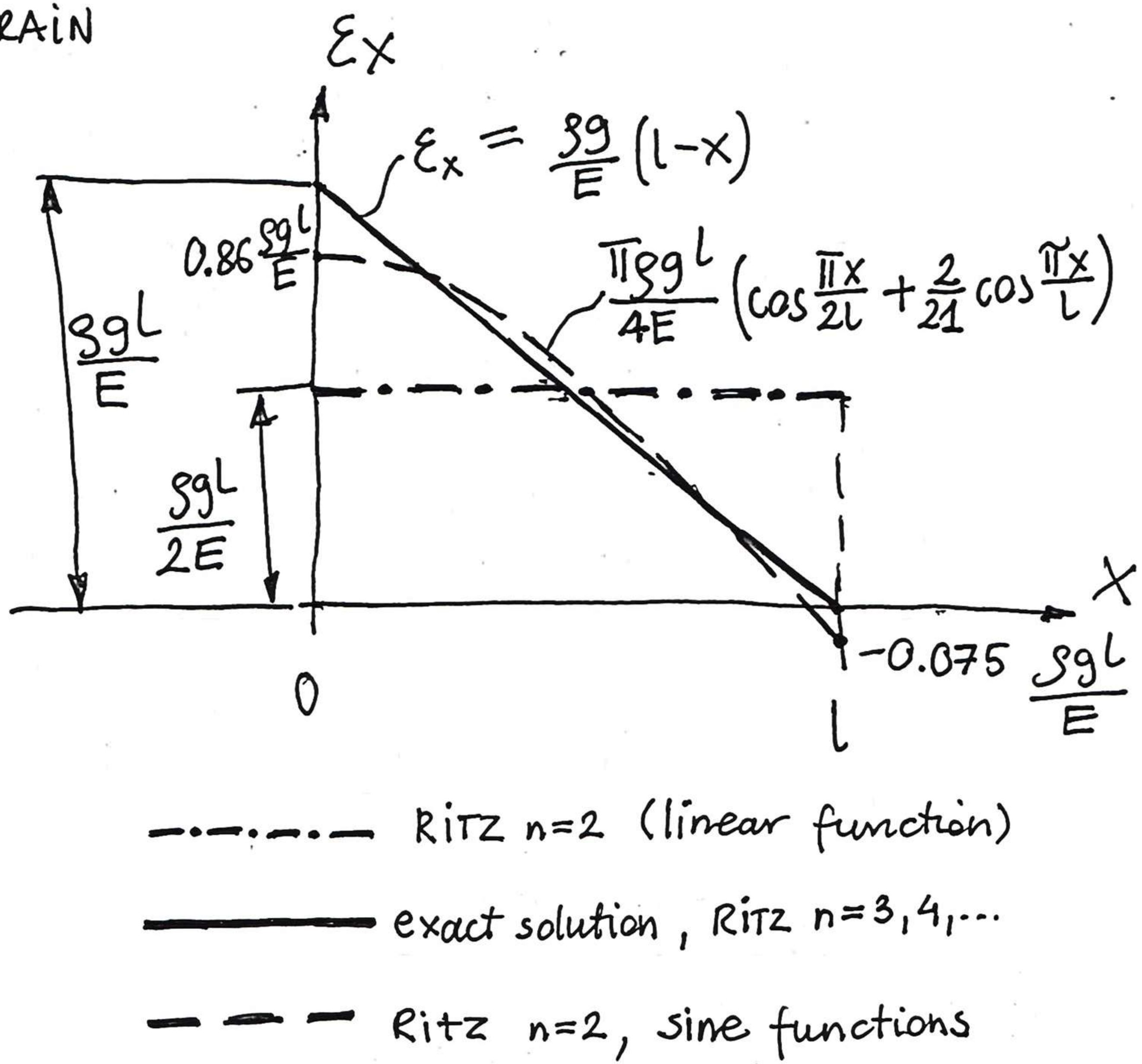


- · - · - · - RITZ n=2 (linear function)

——— exact solution, RITZ n=3,4,...

— — — Ritz n=2, sine functions

STRAIN



ε_x
$\varepsilon_x = \frac{\rho g}{E}(l-x)$
$0.86\frac{\rho g l}{E}$
$\frac{\pi \rho g l}{4E}\left(\cos\frac{\pi x}{2l} + \frac{2}{21}\cos\frac{\pi x}{l}\right)$
$\frac{\rho g l}{E}$
$\frac{\rho g l}{2E}$
x
0
l
$-0.075\,\frac{\rho g l}{E}$
RITZ n=2 (linear function)
exact solution, RITZ n=3,4,...
Ritz n=2, sine functions